Separability of Pure States

$$|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$ad - bc = 0$$

What about higher dimensions?

$$|\Psi\rangle = \sum_{i,j} \psi_{ij}|i,j\rangle$$

Separability of Pure States

$$\rho = |\Psi\rangle\langle\Psi|$$

The state is separable if and only if $\rho^{\Gamma} \geq 0$

First we need a lemma:

If

$$|b\rangle\langle b| = \sum_{i,j} b_i b_j^* |i\rangle\langle j|$$

$$|b\rangle\langle b|^T = \sum_{i,j} b_i b_j^* |j\rangle\langle i|$$

$$|b\rangle\langle b|^T = \sum_{i,j} b_j b_i^* |i\rangle\langle j|$$

Then

$$|b\rangle\langle b|^T = |b^*\rangle\langle b^*|$$

First side:

Assume that $|\Psi\rangle$ is separable:

$$|\Psi\rangle = |a\rangle \otimes |b\rangle$$

$$\rho = |\Psi\rangle\langle\Psi| = |a\rangle\langle a| \otimes |b\rangle\langle b|$$

$$\rho^{\Gamma} = |a\rangle\langle a| \otimes |b^*\rangle\langle b^*|$$

$$\rho^{\Gamma} \geq 0$$

Second side:

Any state has a Schmidt Decomposition:

$$|\Psi\rangle = \sum_{i,j} \lambda_i |i, \alpha_i\rangle$$

$$|\Psi\rangle\langle\Psi| = \sum_{i,j} \lambda_i \lambda_j |i\rangle\langle j| \otimes |\alpha_i\rangle\langle\alpha_j|$$

First consider a simple example:

$$|\Psi\rangle = \lambda_0 |00\rangle + \lambda_1 |11\rangle$$

Second side:

A simple example:

$$|\Psi\rangle = \lambda_0|00\rangle + \lambda_1|11\rangle$$

$$\rho = \begin{pmatrix} \lambda_0^2 & 0 & 0 & \lambda_0\lambda_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_1\lambda_0 & 0 & 0 & \lambda_1^2 \end{pmatrix} \qquad \rho^\Gamma = \begin{pmatrix} \lambda_0^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_0\lambda_1 & 0 \\ 0 & \lambda_0\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$$x = \pm\,\lambda_0\lambda_1$$

So if $|\Psi\rangle$ is entangled, ρ^Γ has negative eigenvalues

Second side: **Another example:**

$$|\Psi\rangle = \lambda_0|00\rangle + \lambda_1|11\rangle + \lambda_2|22\rangle$$

λ_0^2	$\lambda_0\lambda_1$	$\lambda_0\lambda_2$
$\lambda_0\lambda_1$	λ_1^2	$\lambda_1\lambda_2$
$\lambda_0\lambda_2$	$\lambda_1\lambda_2$	λ_2^2

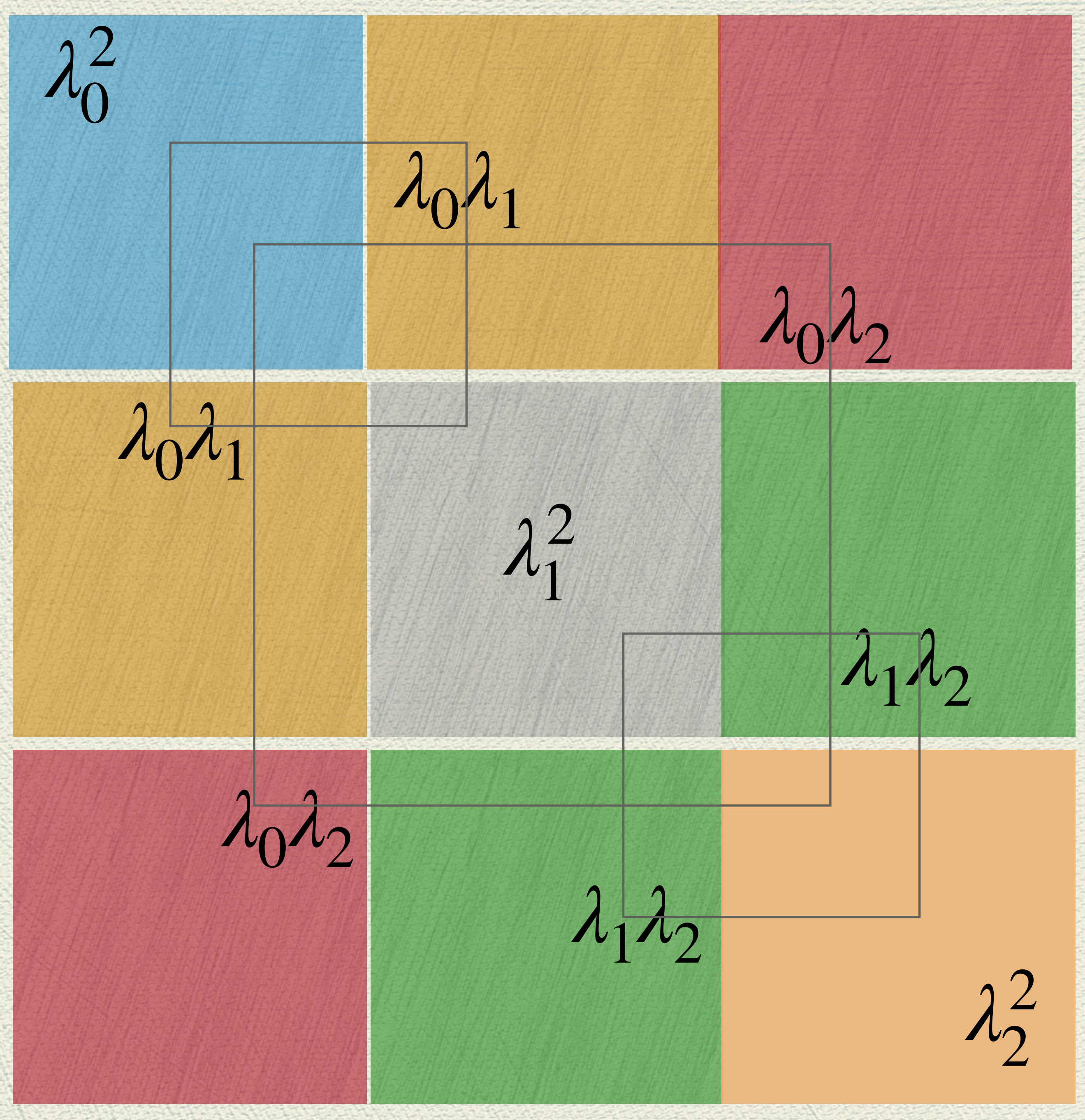
ρ

λ_0^2	$\lambda_0\lambda_1$	$\lambda_0\lambda_2$
$\lambda_0\lambda_1$	λ_1^2	$\lambda_1\lambda_2$
$\lambda_0\lambda_2$	$\lambda_1\lambda_2$	λ_2^2

ρ^Γ

2 by 2 block Structure

ρ^Γ

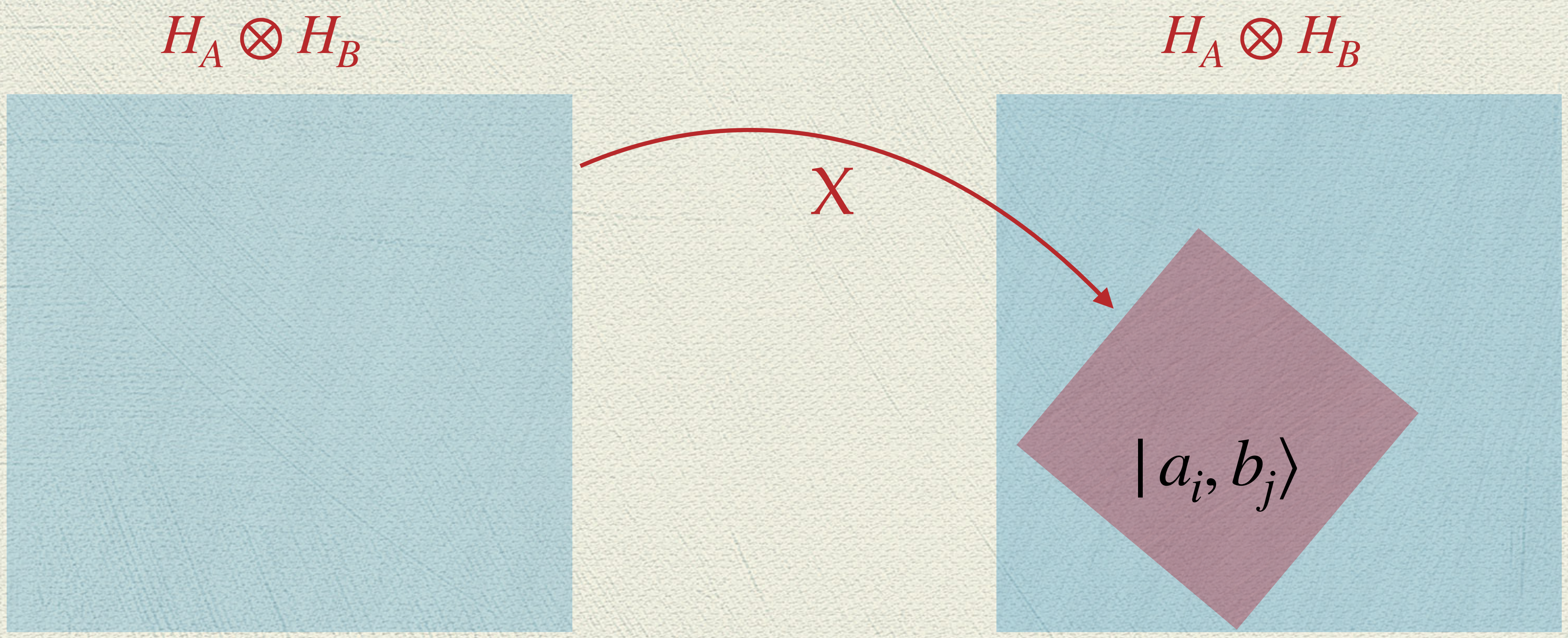


$$x = \pm\,\lambda_0\lambda_1\ ,\ \ \pm\,\lambda_0\lambda_2,\ ,\ \ \pm\,\lambda_1\lambda_2$$

Range Criterion For Separability

1- If X is separable, the range of X is spanned by product states,

$$X = \sum_i p_i \rho_i \otimes \sigma_i$$



Proof: Let X be separable:

$$X = \sum_i p_i \rho_i \otimes \sigma_i$$

$$X = \sum_{i,\alpha,\beta} p_i |a^i_\alpha\rangle\langle a^i_\alpha| \otimes |b^i_\beta\rangle\langle b^i_\beta|$$

$$X|\Psi\rangle = \sum_{i,\alpha,\beta} (\cdots)|a^i_\alpha\rangle|b^i_\beta\rangle$$

$|a^i_\alpha\rangle|b^i_\beta\rangle$ Span Range of X

Now take the partial transpose:

$$X^\Gamma = \sum_{i,\alpha,\beta} p_i |a^i_\alpha\rangle\langle a^i_\alpha| \otimes |b^i_\beta\rangle\langle b^i_\beta|^T$$

Using the lemma:

$$X^\Gamma = \sum_{i,\alpha,\beta} p_i |a^i_\alpha\rangle\langle a^i_\alpha| \otimes |b{*}^i_\beta\rangle\langle b{*}^i_\beta|$$

$$X^\Gamma|\Psi\rangle = \sum_{i,\alpha,\beta} (\cdots)|a^i_\alpha\rangle|b{*}^i_\beta\rangle$$

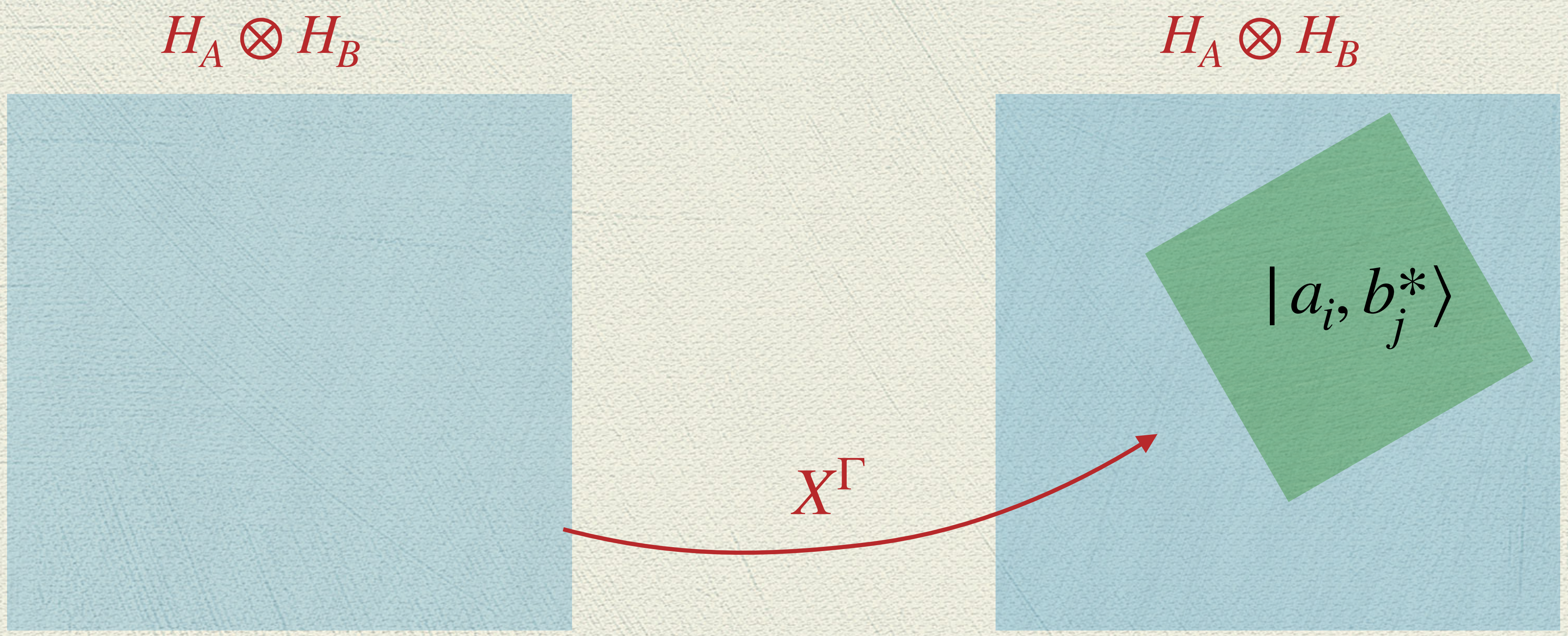
Therefore

$|a^i_\alpha\rangle|b{*}^i_\beta\rangle$ Span Range of X^Γ

And Range Criterion

2- and the range of X^{Γ} is spanned by product states.

$$X = \sum_i p_i \rho_i \otimes \sigma_i$$



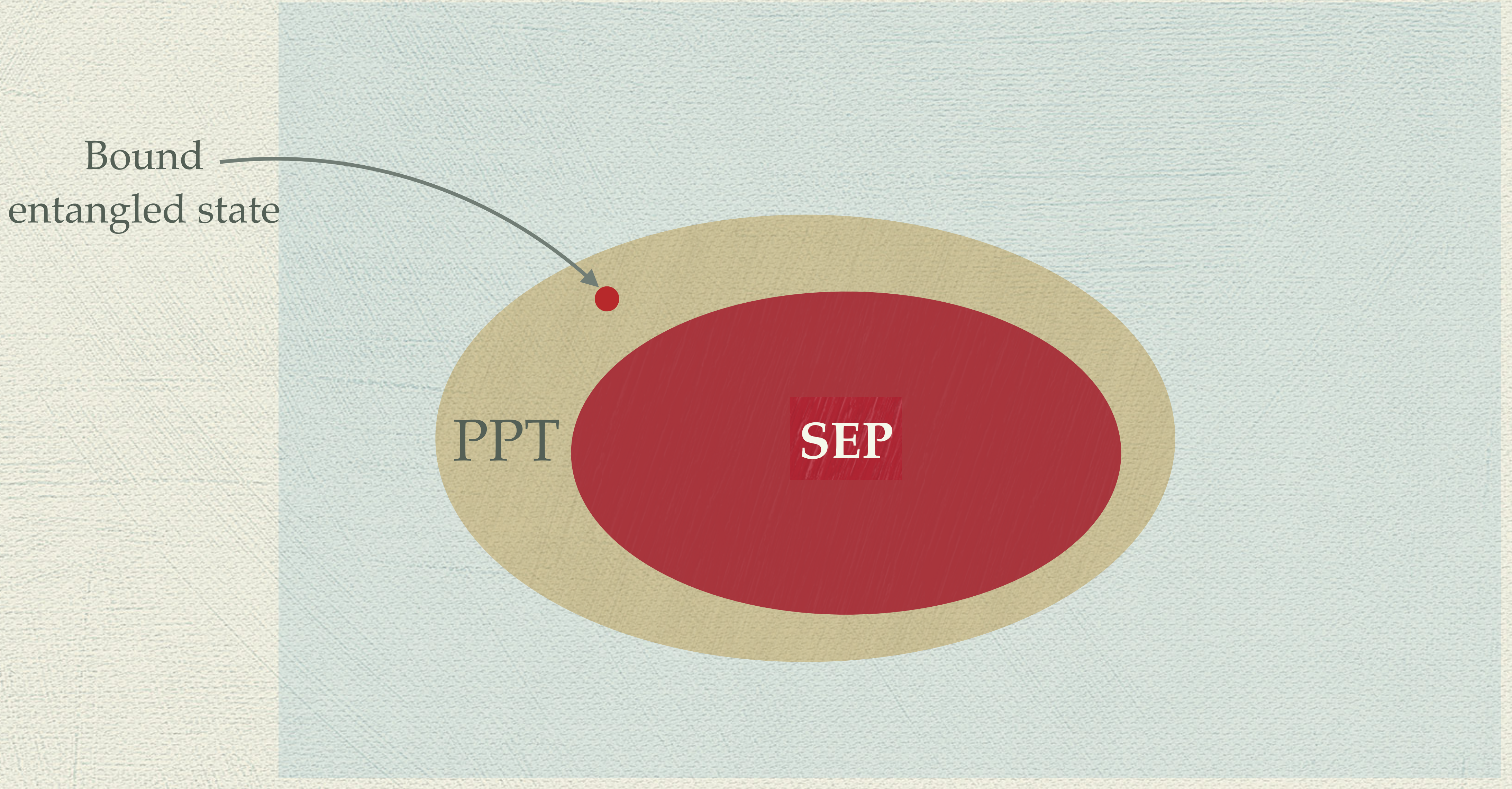
Therefore if X is separable, then

$\exists$ a product basis $\{|a_i, b_j\rangle\}$ which spans the range of X

such that the product basis $\{|a_i, b_j^*\rangle\}$ spans the range of X^Γ

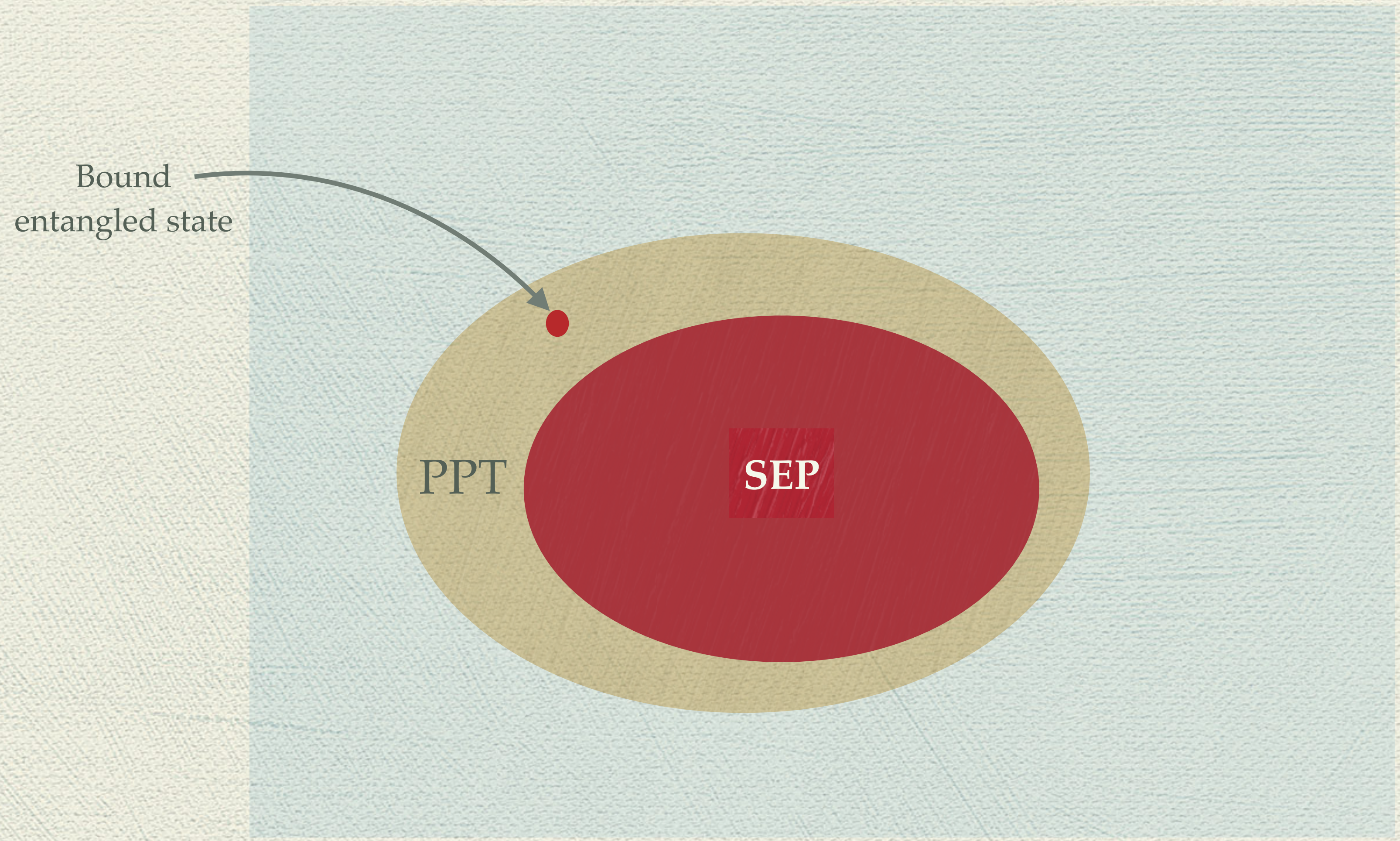
But what is the use of this theorem?

It helps us to construct PPT states



1- Bound Entangled States, cannot be distilled.

2- Bound Entangled States are useless for teleportation.



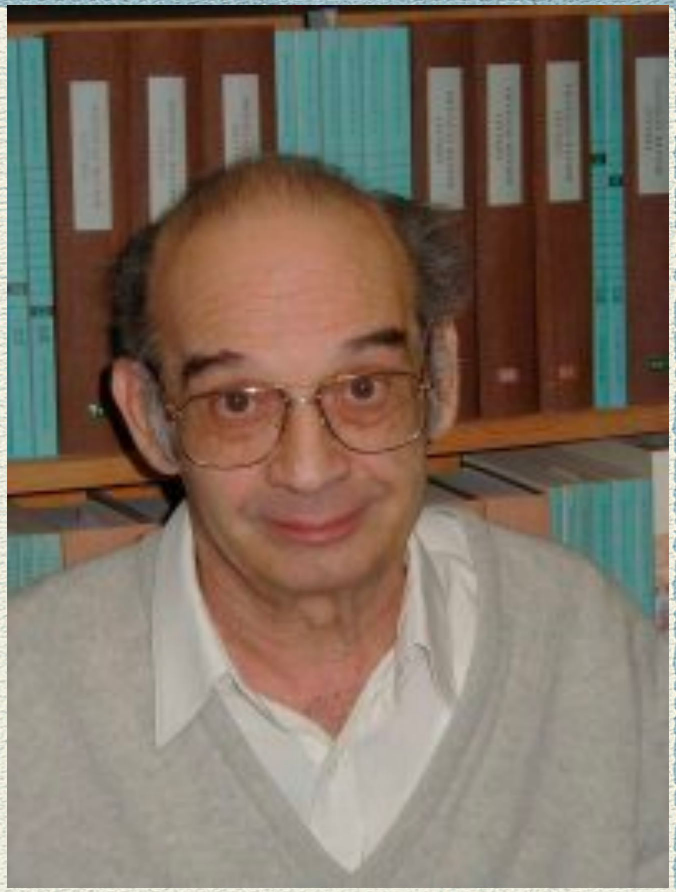
1- Bound Entangled States, cannot be distilled.

2- Bound Entangled States are useless for teleportation.

Peres Conjecture: Bound Entangled States, CANNOT violate Bell inequality.

After a long search

3- Bound Entangled States, CAN violate Bell inequality.



Asher Peres

1934-2005

First we need a new concept

UPB **Un-extendible Product Basis**

$H_A \otimes H_B$

$$\{|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle\}$$

$$\{|++\rangle,\ |+-\rangle,\ |-+\rangle,\ |--\rangle\}$$

$$\{|0+\rangle,\ |0-\rangle,\ |1+\rangle,\ |1-\rangle\}$$

An example of UPB

$H_A \otimes H_B$

A set of product states which cannot be extended to a basis by adding product states.

$$|0\rangle\big(|0\rangle - |1\rangle\big) \qquad |2\rangle\big(|1\rangle - |2\rangle\big)$$

$$\big(|0\rangle - |1\rangle\big)|2\rangle \qquad \big(|1\rangle - |2\rangle\big)|0\rangle$$

$$\big(|0\rangle + |1\rangle + |2\rangle\big)\big(|0\rangle + |1\rangle + |2\rangle\big)$$

$$|0\rangle\big(|0\rangle - |1\rangle\big) \qquad |2\rangle\big(|1\rangle - |2\rangle\big) \qquad \big(|0\rangle - |1\rangle\big)|2\rangle$$

$$\big(|1\rangle - |2\rangle\big)|0\rangle \qquad \big(|0\rangle + |1\rangle + |2\rangle\big)\big(|0\rangle + |1\rangle + |2\rangle\big)$$

$$|\Psi\rangle = \big(a|0\rangle + b|1\rangle + c|2\rangle\big)\big(d|0\rangle + e|1\rangle + f|2\rangle\big)$$

$$a(d-e) = 0$$

$$c(e-f) = 0$$

$$(a-b)f = 0$$

$$(b-c)d = 0$$

$$(a+b+c)(d+e+f) = 0$$

It is not trivial to find UPB

$|0\rangle\big(|0\rangle - |1\rangle\big)$ $\quad |2\rangle\big(|1\rangle - |2\rangle\big)$ $\quad \big(|0\rangle - |1\rangle\big)|0\rangle$

$\big(|1\rangle - |2\rangle\big)|0\rangle$ $\quad \big(|0\rangle + |1\rangle + |2\rangle\big)\big(|0\rangle + |1\rangle + |2\rangle\big)$

$$a(d-e)=0$$

$$c(e-f)=0$$

$$(a-b)d=0$$

$$(b-c)d=0$$

$$(a+b+c)(d+e+f)=0$$

$$|1\rangle\big(|1\rangle - |2\rangle\big)$$

It is not trivial to find UPB, Another Example

$|0\rangle\big(|0\rangle - |1\rangle\big)$ $\quad |1\rangle\big(|1\rangle - |2\rangle\big)$ $\quad \big(|0\rangle - |1\rangle\big)|2\rangle$

$\big(|1\rangle - |2\rangle\big)|0\rangle$ $\quad \big(|0\rangle + |1\rangle + |2\rangle\big)\big(|0\rangle + |1\rangle + |2\rangle\big)$

$$a(d-e)=0$$

$$b(e-f)=0$$

$$(a-b)d=0$$

$$(b-c)d=0$$

$$(a+b+c)(d+e+f)=0$$

$$|2\rangle\big(|1\rangle - |2\rangle\big)$$

An example of Un-extendible Product Basis in 4D

$$|0\rangle\Big(|0\rangle - |1\rangle\Big) \qquad |3\rangle\Big(|1\rangle - |2\rangle\Big) \qquad |2\rangle\Big(|2\rangle - |3\rangle\Big)$$

$$\Big(|0\rangle - |1\rangle\Big)|3\rangle \qquad \Big(|1\rangle - |2\rangle\Big)|0\rangle \qquad \Big(|2\rangle - |3\rangle\Big)|1\rangle$$

$$\Big(|0\rangle + |1\rangle + |2\rangle + |3\rangle\Big)\Big(|0\rangle + |1\rangle + |2\rangle + |3\rangle\Big)$$

We use a UPB and construct a bound entangled state.

$$X = I \otimes I - \sum_i |\alpha_i, \beta_i\rangle\langle\alpha_i, \beta_i|$$

UPB

$$X^\Gamma = I \otimes I - \sum_i |\alpha_i, \beta_i^*\rangle\langle\alpha_i, \beta_i^*|$$

1-X is a positive operator

2-X is PPT

3-X and X^Γ are projections on entangled states,

$$X = I \otimes I - \sum_i |\alpha_i, \beta_i\rangle\langle\alpha_i, \beta_i|$$

UPB

So any such X is a bound entangled state.

Examples of Entanglement Witnesses

A new definition

$Tr(W\rho_{sep}) \geq 0$ For any Separable State

$$\langle a, b | W | a, b \rangle \geq 0 \qquad \forall \; |a, b\rangle$$

W is Block – Positive

But is not positive.

Let A and B be positive operators.

$W = A \otimes B$ is obviously block-positive

It is also positive.

$$\lambda_{ij}(W) = \lambda_i(A)\lambda_j(B)$$

Let A, B, C and D be positive operators.

$W = A \otimes B + C \otimes D$ is block-positive, since

$$\langle x, y | W | x, y \rangle = \langle x, y | A \otimes B | x, y \rangle + \langle x, y | C \otimes D | x, y \rangle$$

But it is not necessarily positive.

This shows us how to construct EW's.

An obvious Witness: The SWAP operator

$$P = \sum_{i,j} |i, j\rangle\langle j, i|$$

$$\langle a, b | P | a, b\rangle = \langle a, b | b, a\rangle = |\langle a | b\rangle|^2$$

$$P(|a, b\rangle - |b, a\rangle) = |b, a\rangle - |a, b\rangle$$

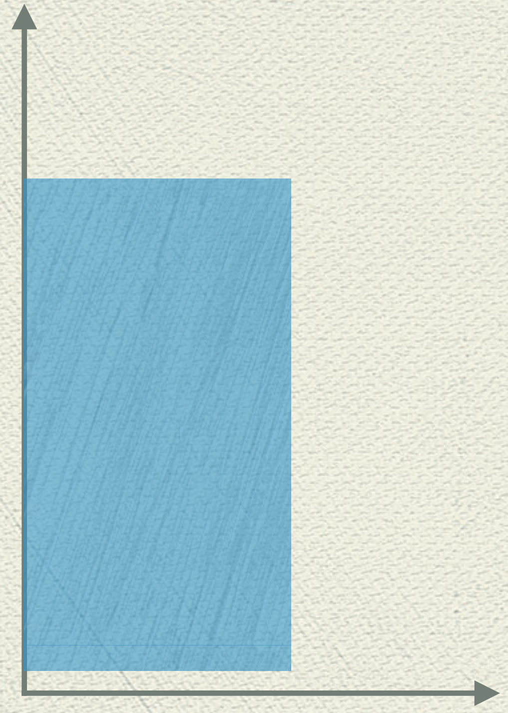
How to construct Block-positive operators?

Let $P \geq 0$

Define: $W_\lambda = P - \lambda I \otimes I$

Spectrum of P

Spectrum of W_λ



Let $P \geq 0$ $\quad W_\lambda = P - \lambda I \otimes I$

$$\langle a, b | W_\lambda | a, b \rangle = \langle a, b | P | a, b \rangle - \lambda \geq 0$$

$$\lambda \leq \langle a, b | P | a, b \rangle$$

$$\lambda \leq \underset{|a,b\rangle}{Inf} \langle a, b | P | a, b \rangle$$

Werner States

$$\rho_w := \alpha Q_S + \beta Q_A$$

Q_S Projector onto the symmetric subspace

Q_A Projector onto the anti-symmetric subspace

$$\rho_w := \frac{f}{d_-} Q_A + \frac{1-f}{d_+} Q_S$$

Werner States in two dimensions

$$\rho_w := \frac{f}{d_-} Q_A + \frac{1-f}{d_+} Q_S$$

$$Q_A = |\psi^-\rangle\langle\psi^-|$$

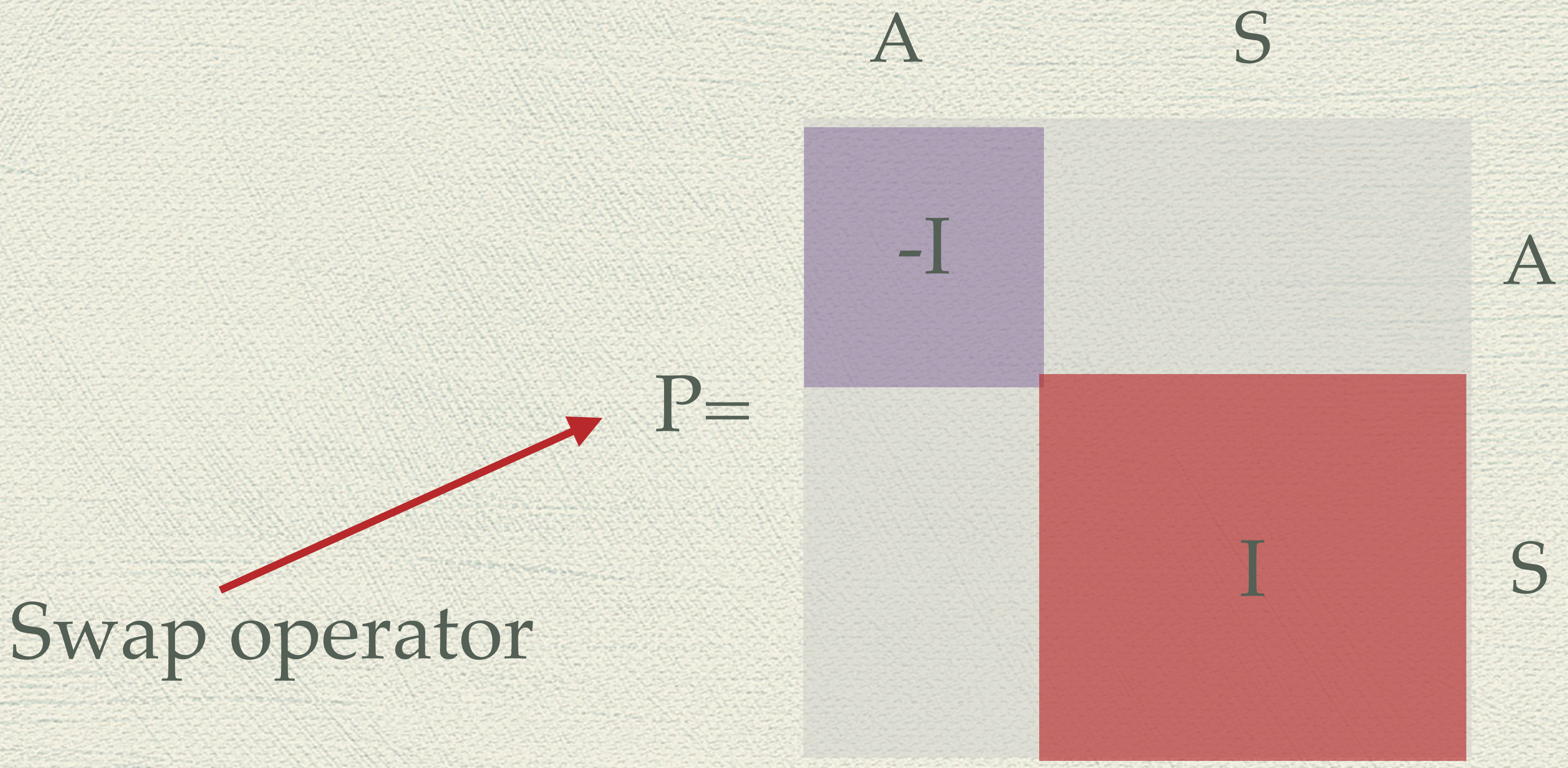
$$Q_S = I - |\psi^-\rangle\langle\psi^-|$$

$$\rho_w = f\,|\psi^-\rangle\langle\psi^-| + \frac{1-f}{3}(I - |\psi^-\rangle\langle\psi^-|)$$

Or

$$\rho_w = g\,|\psi^-\rangle\langle\psi^-| + \frac{1-g}{2} I$$

Question: For what value of f, the Werner state is entangled?



$$Tr(PQ_A) = -\, dim(H_A) = -\, d_-$$

$$Tr(PQ_S) = dim(H_S) = d_+$$

Question: For what value of f, the Werner state is entangled?

$$\rho_w = \frac{f}{d_-} Q_A + \frac{1-f}{d_+} Q_S$$

$$Tr(\rho_w P) = \frac{f}{d_-}(-d_-) + \frac{1-f}{d_+} d_+ = 1 - 2f$$

If $f > \frac{1}{2}$ → ρ_w is entangled

Re-alignment Criterion

Schmidt Decomposition

$$\rho = \sum_i \lambda_i \, A_i \otimes B_i$$

$$\langle A_i | A_j \rangle = \langle B_i | B_j \rangle = \delta_{ij}$$

Theorem:

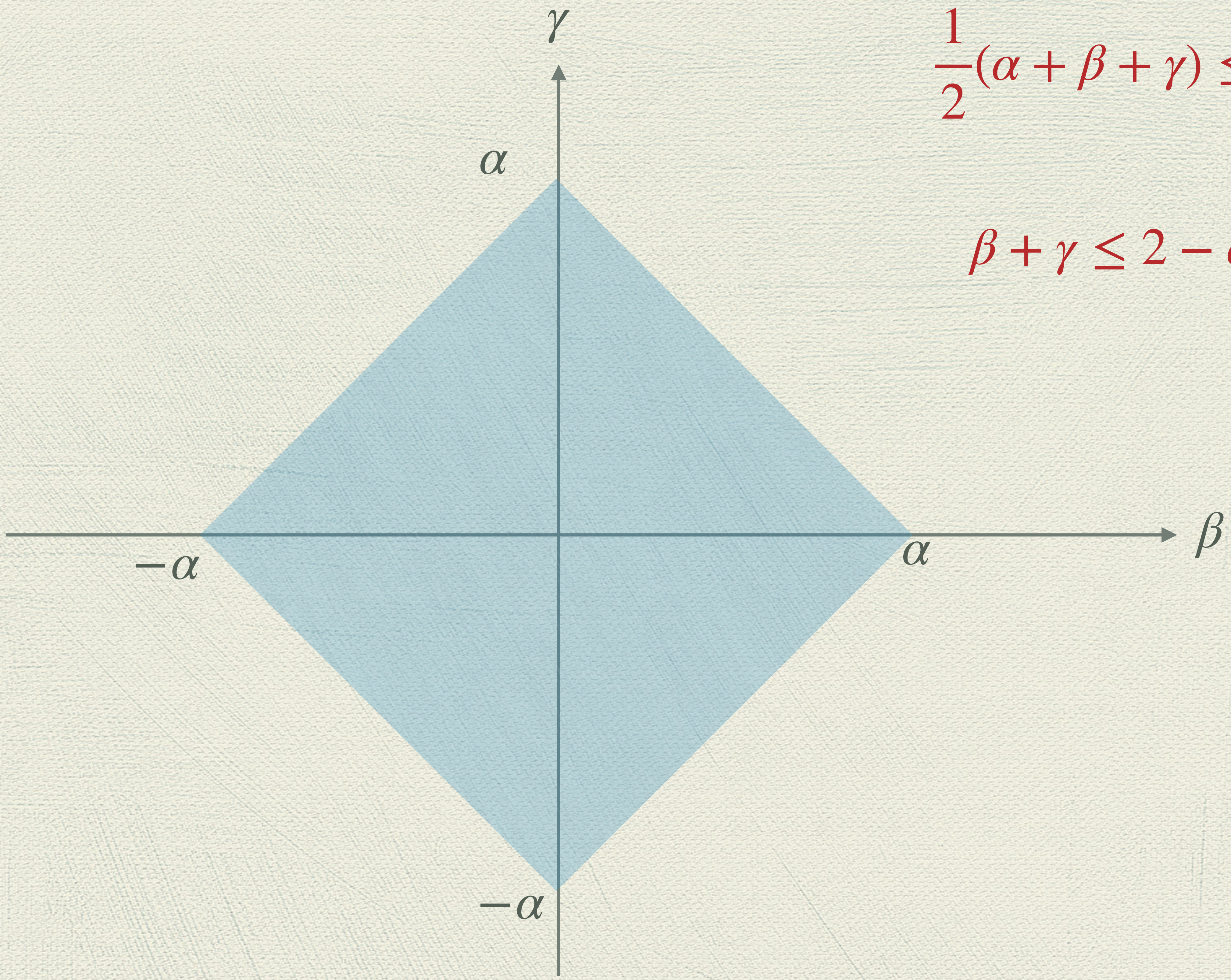
Chen and Wu (2003), Rudolf (2005)

If $\sum_i \lambda_i < 1$, ρ is Separable

Re-alignment Criterion

$$\langle \sigma_i \sigma_j \rangle = 2\delta_{ij}$$

$$\rho = \frac{1}{4}\left[\alpha I \otimes + \beta \sigma_x \otimes \sigma_x + \gamma \sigma_y \otimes \sigma_y\right]$$



$$\frac{1}{2}(\alpha + \beta + \gamma) \leq 1$$

$$\beta + \gamma \leq 2 - \alpha$$

How to make EW from Re-alignment Criterion

$$W := I \otimes I - \sum_i A_i \otimes B_i$$

Question 1: Is W block-positive?

Question 2: Does W have a negative eigenvalue?

Question 1: Is W block-positive?

$$\langle \phi, \psi | W | \phi, \psi \rangle = 1 - \sum_i \langle \phi | A_i | \phi \rangle \langle \psi | B_i | \psi \rangle$$

Since A_i and B_i are bases:

$$|\phi\rangle\langle\phi| = \sum_i \gamma_i A_i \qquad |\psi\rangle\langle\psi| = \sum_i \delta_i B_i$$

$$\langle \phi | A_i | \phi \rangle = Tr(|\phi\rangle\langle\phi| A_i) = \gamma_i$$

$$\langle \psi | B_i | \psi \rangle = Tr(|\psi\rangle\langle\psi| B_i) = \delta_i$$

$$1 = \langle \phi | \phi \rangle = Tr(|\phi\rangle\langle\phi|) = Tr(|\phi\rangle\langle\phi|^2)$$

$$= \sum_{i,j} Tr(\gamma_i^* \gamma_j A_i^\dagger A_j) = \sum_i |\gamma_i|^2$$

$$\langle \phi | A_i | \phi \rangle = Tr(|\phi\rangle\langle\phi| A_i) = \gamma_i$$

$$\langle \psi | B_i | \psi \rangle = Tr(|\psi\rangle\langle\psi| B_i) = \delta_i$$

$$\sum_i \langle \phi | A_i | \phi \rangle \langle \psi | B_i | \psi \rangle = \sum_i \gamma_i \delta_i \leq \sqrt{\|\gamma\| \; \|\delta\|} = 1$$

$$\langle \phi, \psi | W | \phi, \psi \rangle = 1 - \sum_i \langle \phi | A_i | \phi \rangle \langle \psi | B_i | \psi \rangle \geq 0$$

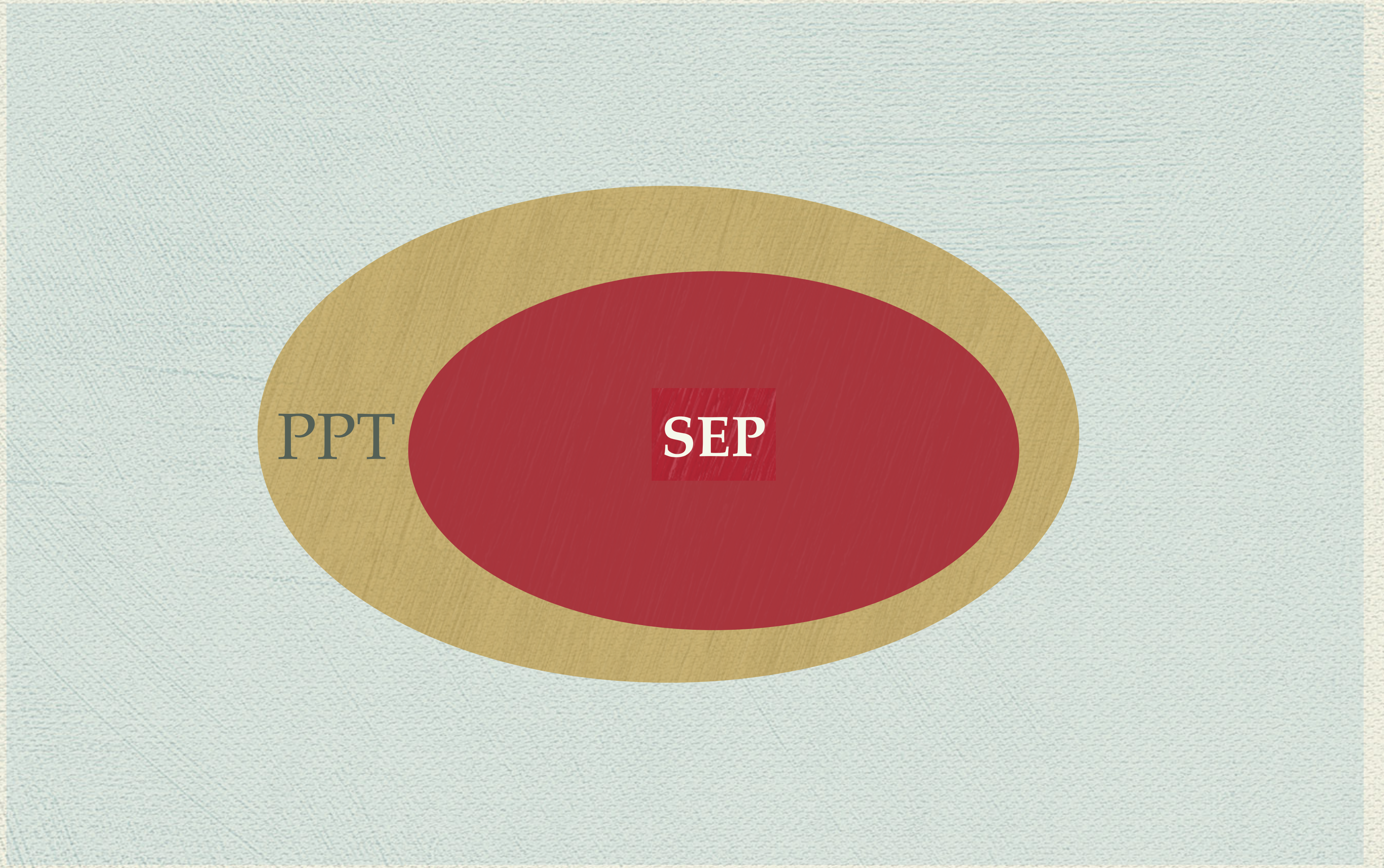
Question 2: Does W have a negative eigenvalue?

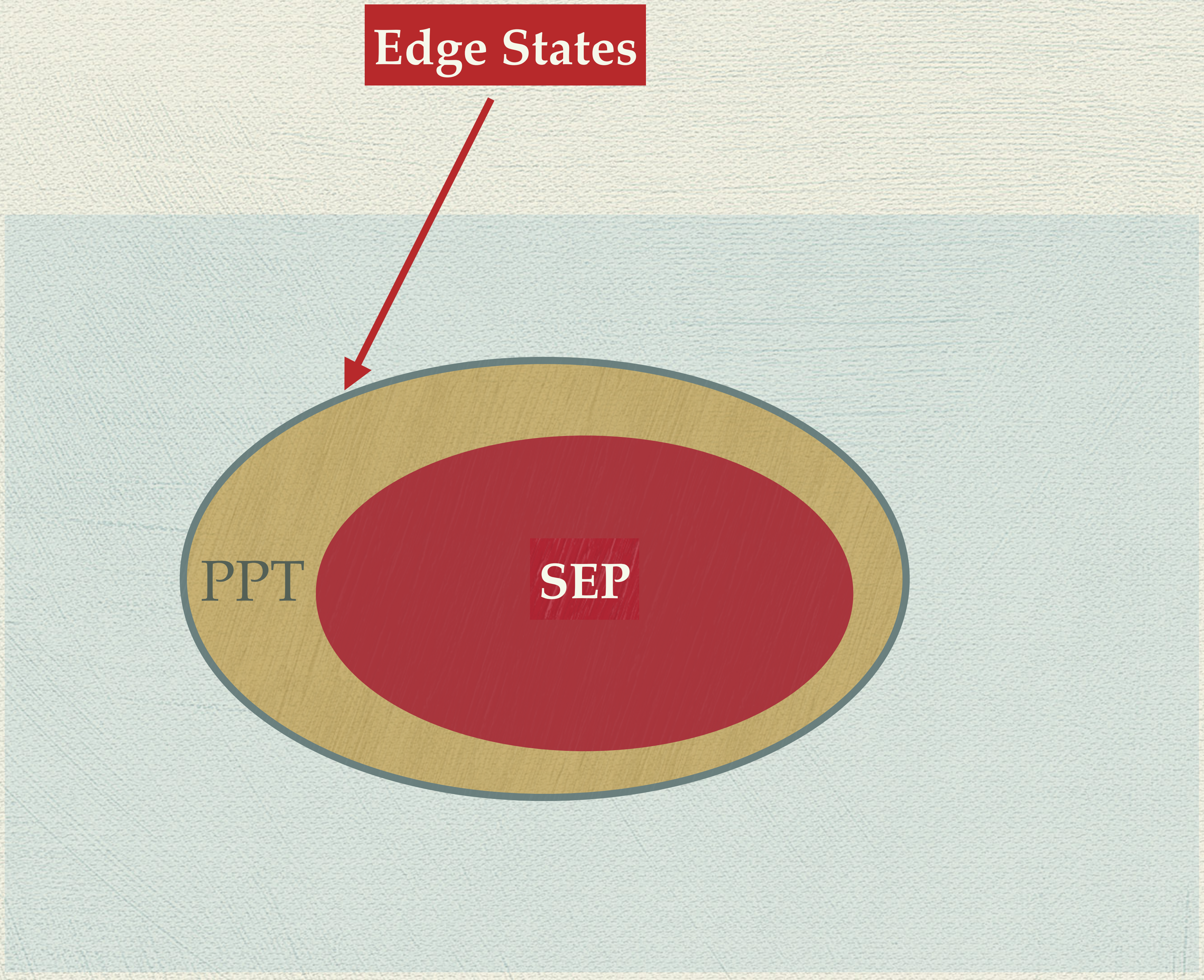
A few examples:

$$W = I \otimes I - \frac{1}{2}\sigma_x \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \geq 0$$

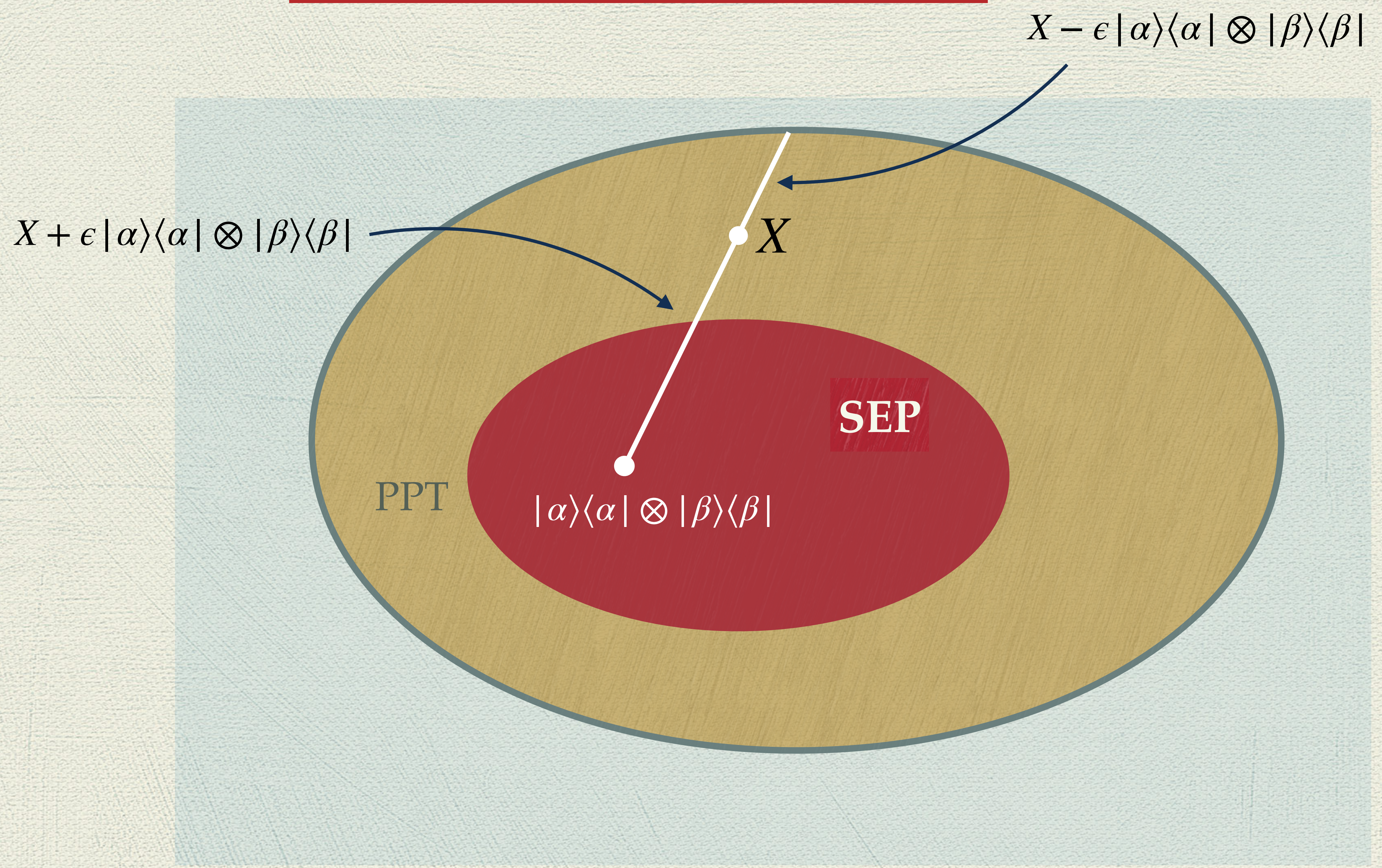
$$W = I \otimes I - \frac{1}{2}\sigma_x \otimes \sigma_x - \frac{1}{2}\sigma_y \otimes \sigma_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \geq 0$$

Edge States

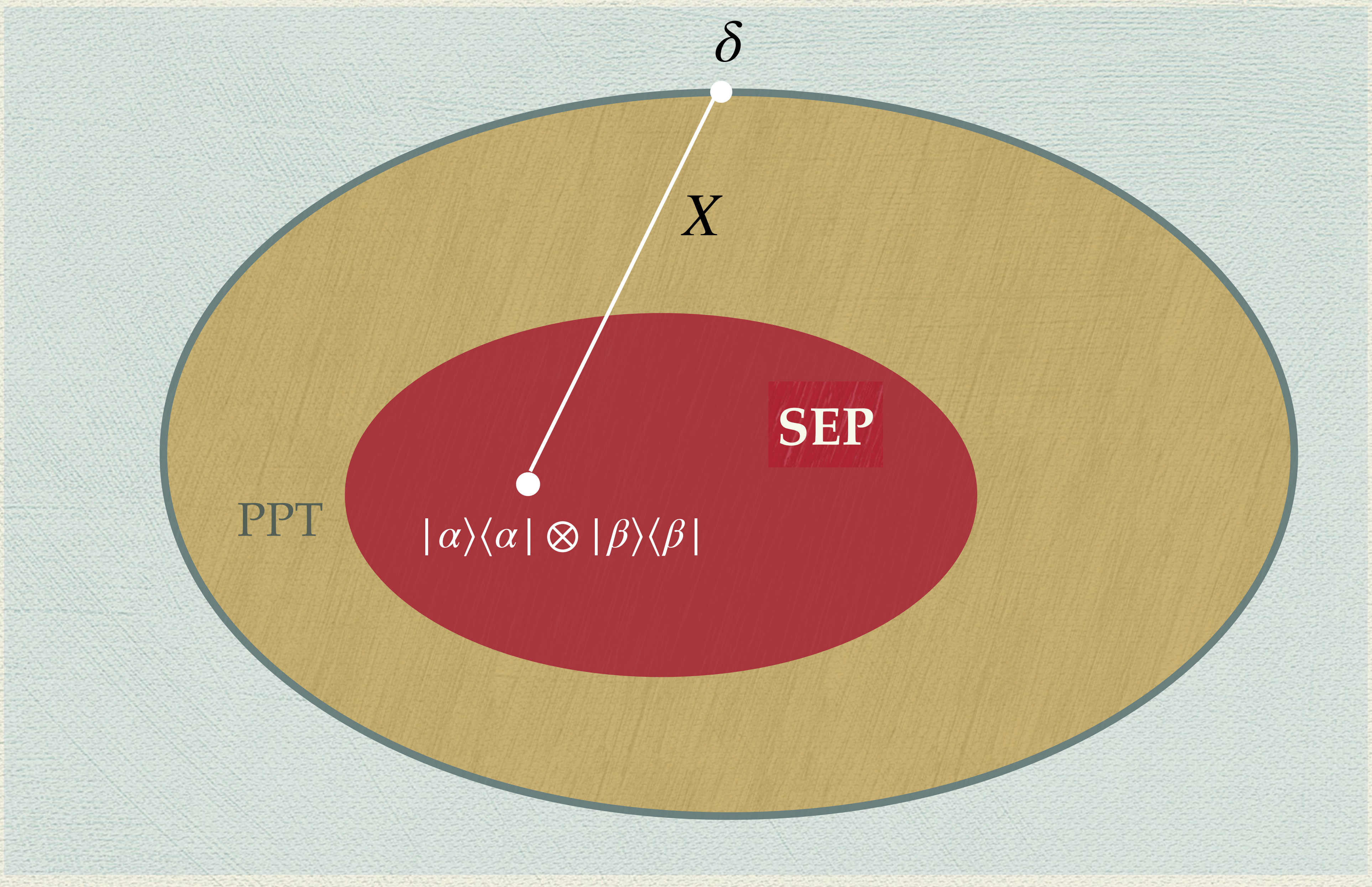



Edge States
PPT
SEP

How to characterize Edge States?



How to characterize Edge States?



δ is an edge state if for any $|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$

and any ϵ

$\delta - \epsilon|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$ is not PPT

Decomposable Witness

$$W = P + Q^{\Gamma}$$

What is the form of indecomposable Witness?

$$W = P + Q^{\Gamma} - aI$$

How large the value of a can be?

We want:

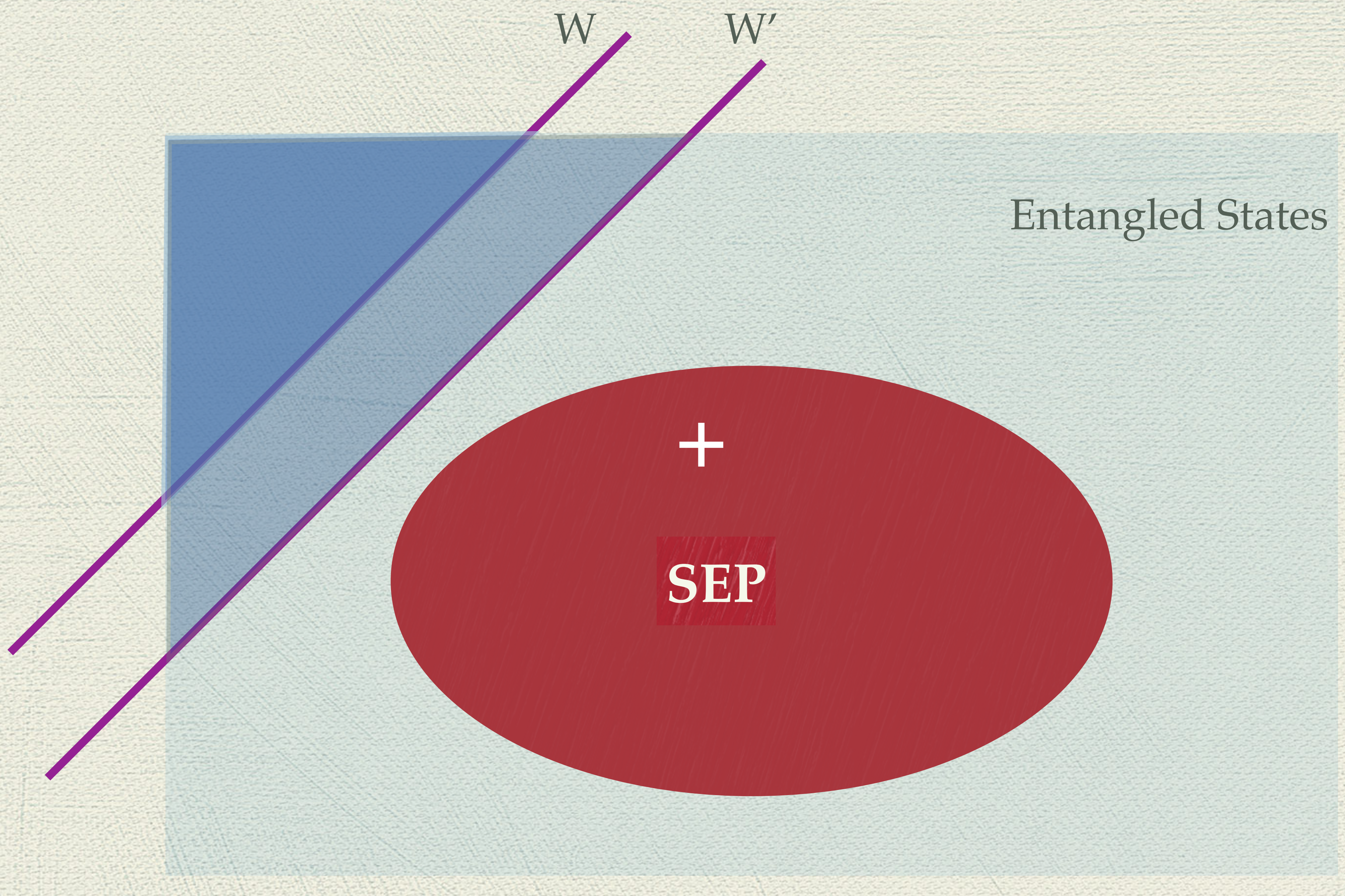
$$\langle \alpha, \beta \,|\, W \,|\, \alpha, \beta \rangle \geq 0$$

$$\langle \alpha, \beta \,|\, P + Q^{\Gamma} - \epsilon I \,|\, \alpha, \beta \rangle \geq 0$$

$$\langle \alpha, \beta \,|\, P + Q^{\Gamma} \,|\, \alpha, \beta \rangle \geq \epsilon$$

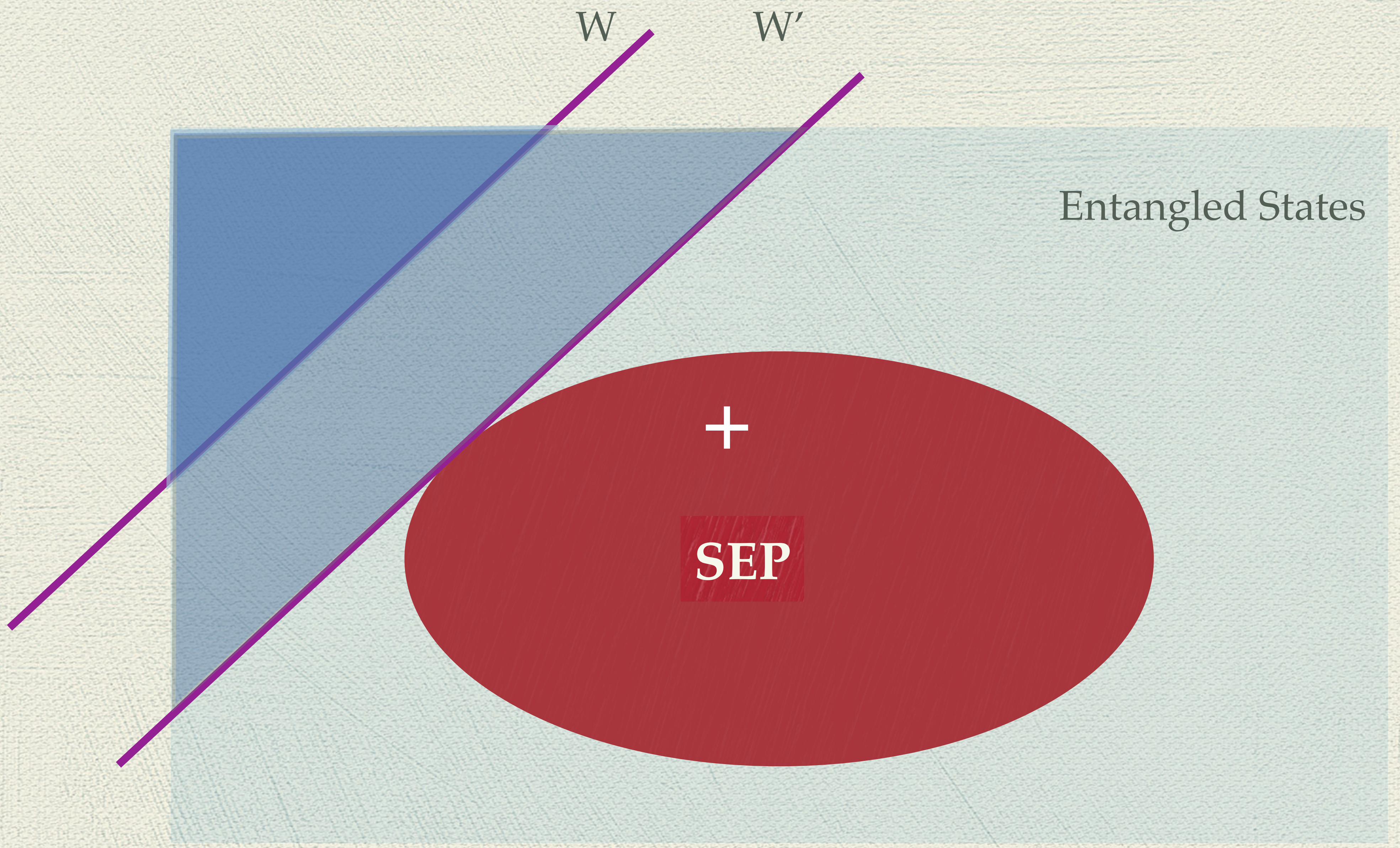
$$0 \leq \epsilon \leq \underset{|\alpha,\beta\rangle}{Inf} \; \langle \alpha, \beta \,|\, P + Q^{\Gamma} \,|\, \alpha, \beta \rangle \,.$$

How to construct Optimal Witnesses



W′ is finer than W.

Theorem: W is optimal if for any positive P, W-P is no longer Block-Positive.



Any optimal Witness is of the form Q^Γ for some Q ≥ 0

$$Q = \sum_i |\phi_i\rangle\langle\phi_i|$$

$$W = Q^\Gamma = \sum_i |\phi_i\rangle\langle\phi_i|^\Gamma$$

Extremal Witnesses

The SWAP operator is an optimal witness.

$$|\phi^+\rangle\langle\phi^+| = \sum_{i,j} |i,i\rangle\langle\, |j,j|$$

$$|\phi^+\rangle\langle\phi^+|^\Gamma = \sum_{i,j} |i,j\rangle\langle j,i| = P$$

End of Part II